

# Chance Constrained Model Predictive Control

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### **Abstract**

*This work focuses on robustness of model predictive control (MPC) with respect to satisfaction of process output constraints. A method of improving such robustness is presented. The method relies on formulating output constraints as chance constraints using the uncertainty description of the process model. The resulting on-line optimization problem is convex. The proposed approach is illustrated through a simulation case study on a high-purity distillation column. Suggestions for further improvements are made.*

## **Introduction**

Robustness is a highly desirable property for process control systems. Qualitatively speaking, a controller is robust if it results in actual closed-loop behavior that does not deviate unacceptably from the nominal closed-loop behavior, which, in turn, corresponds to a nominal process behavior. For example, a model-based controller results in robust closed-loop stability if the closed loop is stable even if there is a discrepancy between the model used by the controller and the actual process behavior. The extent of such discrepancy for which closed-loop stability is maintained corresponds to the degree of robustness of that controller. Although necessary, robust stability is usually not sufficient for good controller performance. Other closed-loop properties may have to be maintained in the presence of discrepancy between the nominal behavior of a process and its actual one. For instance, the resulting regulation error magnitude (e.g. its 2-norm or  $\infty$ -norm) in a feedback loop has to remain “small” in the presence of nominal/actual process behavior discrepancy. Such a requirement is frequently referred to as robust performance. Along with robust stability, robust performance, as defined in the previous sentence, has been studied extensively. However, as explained above, there are many more properties that capture closed-loop performance. One such property, particularly important for constrained model predictive control (MPC) systems, is the satisfaction of various inequality constraints.

Inequality constrained MPC systems rely on the on-line optimization of an objective function over a future moving horizon, subject to various constraints. At each time step, process measurements are used to formulate the on-line optimization problem whose solution determines an optimal input, which is fed to the process.

The robustness of unconstrained MPC has been studied extensively. Since an unconstrained MPC system is equivalent to a linear time-invariant system, robust unconstrained MPC analysis and synthesis methods relying on either time-domain or frequency-domain descriptions can be used. Discussions of frequency-domain and time-domain methods can be found in Morari and Zafiriou (1989) and Mosca (1995), respectively. For constrained MPC systems, the study of robustness has several facets, and is at a less mature stage. Robust stability results for constrained MPC, within the framework set by Rawlings and Muske (1993), have been presented by a number of investigators, including Genceli and Nikolaou (1993), Michalska and Mayne (1993), Zheng and Morari (1993), Chen and Allgöwer (1996), Lee and Yu (1997), Badgwell (1997), De Nicolao et al. (1998). The purpose of this work is to examine a different aspect of constrained MPC robustness, namely robustness with respect to satisfaction by the actual system of inequality constraints posed in the on-line optimization problem. While inequality constraints that place bounds on process inputs can be easily satisfied by the actual system, constraints on process outputs are more elusive. That is because future process outputs within an MPC moving horizon have to be predicted on the basis of a process model (involving the effects of manipulated inputs and disturbances on process outputs). Because the model involves uncertainty, process output predictions are also uncertain. This uncertainty in process output predictions may result in adverse violation of output constraints by the actual closed-loop system, even though predicted outputs over the moving horizon might have been properly constrained. Consequently, a method of incorporating model uncertainty into the output constraints of the on-line optimization is needed. This would improve the robustness of constrained MPC. In this paper, we introduce an approach towards achieving that goal.

The proposed approach relies on formulating output constraints of the type  $y_{\min} \leq y \leq y_{\max}$  as chance constraints of the type

$$(1) \quad \Pr\{y_{\min} \leq y \leq y_{\max}\} \geq \alpha$$

where  $\Pr\{A\}$  is the probability of event  $A$  occurring,  $y$  is the process output bounded by  $y_{\min}$  and  $y_{\max}$ , and  $\alpha$  is the specified probability, or confidence level, that the output constraint would be satisfied. Under the assumption that the process output  $y$  is predicted by a linear model with normally distributed coefficients, the above chance constraint can be reformulated as a convex, deterministic constraint on process inputs. This new constraint can then be readily incorporated into the standard MPC formulation. The resulting on-line optimization problem can be solved using reliable convex optimization algorithms.

The rest of the paper is structured as follows: We first provide a brief overview of stochastic programming and chance-constraint optimization. Next, we show how the MPC on-line optimization problem can be cast as a chance constraint problem. Subsequently, we present an example of using chance-constrained MPC on a high-purity distillation column, an ill-conditioned system. Finally, we draw conclusions and make suggestions for further research.

## Stochastic Programming and Chance-Constraint Optimization

Stochastic programming is an optimization technique in which the constraints or objective function of an optimization problem contain stochastic parameters. Chance-constrained optimization is one method of stochastic programming that attempts to reconcile optimization over uncertain constraints. The constraints, which contain stochastic parameters, are guaranteed to be satisfied with a certain probability at the optimum found. A typical chance constrained stochastic programming problem has the following form (Birge and Louveaux, 1997):

$$(2) \quad \begin{aligned} & \min_{\mathbf{x}} f(\mathbf{x}) \\ & \text{s.t. } \mathbf{g}_1(\mathbf{x}) \leq \mathbf{0} \\ & \quad \Pr\{\mathbf{g}_2(\mathbf{p}, \mathbf{x}) \leq \mathbf{0}\} \geq \alpha \end{aligned}$$

where  $\mathbf{x} \in \mathfrak{R}^n$  is the decision variable vector,  $f(\mathbf{x}) \in \mathfrak{R}$ ,  $\mathbf{g}_1(\mathbf{x}) \in \mathfrak{R}^{m_1}$ , and  $\mathbf{g}_2(\mathbf{p}, \mathbf{x}) \in \mathfrak{R}^{m_2}$  contains the stochastic parameter vector  $\mathbf{p} \in \mathfrak{R}^p$ . If the probability density function of  $\mathbf{p}$  is known, then the probabilistic constraint  $\Pr\{\mathbf{g}_2(\mathbf{p}, \mathbf{x}) \leq \mathbf{0}\} \geq \alpha$  can, in principle, be substituted by a deterministic constraint of the form  $\mathbf{g}_3(\mathbf{x}) \leq \mathbf{0}$ , so that the entire optimization problem can be handled as an ordinary nonlinear programming problem.

Depending on the form of  $\mathbf{g}_2$ , the explicit form of  $\mathbf{g}_3$  may be difficult to obtain. The task of developing an explicit closed form for  $\mathbf{g}_3$  is greatly simplified if  $\mathbf{g}_2$  is affine in the parameter vector  $\mathbf{p}$ , i.e.  $\mathbf{g}_2(\mathbf{p}, \mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{p} + \mathbf{b}(\mathbf{x})$ , where  $\mathbf{A}(\mathbf{x}) \in \mathfrak{R}^{m_2 \times p}$ .

For situations in which the stochastic parameters  $\mathbf{p}$  can be separated from the decision variable  $\mathbf{x}$  in a constraint such as

$$(3) \quad \Pr\{\tilde{\mathbf{g}}_2(\mathbf{x}) \leq \mathbf{p}\} \geq \alpha$$

the deterministic equivalent can be found fairly easily by using the probability density function (pdf) of  $\mathbf{p}$  to find a value,  $\beta$ , from

$$(4) \quad \alpha = \int \cdots \int_{\beta}^{\infty} pdf(\mathbf{p}) d\mathbf{p}$$

such that if  $\tilde{\mathbf{g}}_2(\mathbf{x}) \leq \beta$ , then eqn. (3) holds.

Linearly constrained problems with normally distributed stochastic parameters can also be handled efficiently. Consider, for example, a constraint of the form

$$(5) \quad \Pr\{\mathbf{a}^T(\mathbf{M}\mathbf{x} + \mathbf{c}) \leq b\} \geq \alpha$$

where  $\mathbf{a}$  is a normally distributed vector with mean  $\bar{\mathbf{a}}$  and covariance  $\mathbf{P}_a$ . Since  $\mathbf{a}^T(\mathbf{M}\mathbf{x} + \mathbf{c})$  is a linear transformation of  $\mathbf{a}$ , then it is normally distributed. The mean of  $\mathbf{a}^T(\mathbf{M}\mathbf{x} + \mathbf{c})$  is  $\bar{\mathbf{a}}^T(\mathbf{M}\mathbf{x} + \mathbf{c})$  and its variance

$$(6) \quad \text{Var}(\mathbf{a}^T(\mathbf{M}\mathbf{x} + \mathbf{c})) = (\mathbf{M}\mathbf{x} + \mathbf{c})^T \mathbf{P}_a (\mathbf{M}\mathbf{x} + \mathbf{c})$$

Using eqn. (6) we can then rearrange the constraint in eqn. (5) to the standard normal form

$$(7) \quad \Pr \left\{ \frac{\mathbf{a}^T (\mathbf{M}\mathbf{x} + \mathbf{c}) - \bar{\mathbf{a}}^T (\mathbf{M}\mathbf{x} + \mathbf{c})}{\sqrt{(\mathbf{M}\mathbf{x} + \mathbf{c})^T \mathbf{P}_a (\mathbf{M}\mathbf{x} + \mathbf{c})}} \leq \frac{b - \bar{\mathbf{a}}^T (\mathbf{M}\mathbf{x} + \mathbf{c})}{\sqrt{(\mathbf{M}\mathbf{x} + \mathbf{c})^T \mathbf{P}_a (\mathbf{M}\mathbf{x} + \mathbf{c})}} \right\} \geq \alpha$$

Then using the value of the confidence level,  $\alpha$ , we get

$$(8) \quad \frac{b - \bar{\mathbf{a}}^T (\mathbf{M}\mathbf{x} + \mathbf{c})}{\sqrt{(\mathbf{M}\mathbf{x} + \mathbf{c})^T \mathbf{P}_a (\mathbf{M}\mathbf{x} + \mathbf{c})}} \geq K_\alpha$$

where  $K_\alpha$  is the value of the inverse cumulative distribution function of the standard normal distribution evaluated at  $\alpha$ , usually denoted as  $F^{-1}(\alpha)$ . Hence the stochastic constraint of eqn. (5) can be recast as the deterministic constraint of eqn. (8). Note that eqn. (8), rewritten as

$$(9) \quad \bar{\mathbf{a}}^T (\mathbf{M}\mathbf{x} + \mathbf{c}) \leq b - K_\alpha \sqrt{(\mathbf{M}\mathbf{x} + \mathbf{c})^T \mathbf{P}_a (\mathbf{M}\mathbf{x} + \mathbf{c})},$$

indicates that merely replacing  $\mathbf{a}$  by  $\bar{\mathbf{a}}$  in eqn. (5) and then replacing eqn. (5) by the deterministic inequality

$$(10) \quad \bar{\mathbf{a}}^T (\mathbf{M}\mathbf{x} + \mathbf{c}) \leq b$$

is not correct, because it can lead to violation of the constraint

$$(11) \quad \mathbf{a}_r^T (\mathbf{M}\mathbf{x} + \mathbf{c}) \leq b$$

where  $\mathbf{a}_r$  is a realization of the random variable  $\mathbf{a}$ , with high probability. As will become clear in the sequel, this observation is important for MPC systems employing uncertain models in which parameters appear linearly (e.g., linear, Volterra, Hammerstein, Wiener, etc.) and output inequality constraints.

## Chance-Constrained MPC

Consider a process whose output  $y$  must stay below an upper bound  $y_{\max}$ . That requirement would normally be translated into a set of MPC constraints of the form

$$(12) \quad y(k+i|k) \leq y_{\max}, \quad i=1, \dots, n_c$$

that would have to be incorporated in an optimization problem solved at time  $k$ . Process output constraints such as in eqn. (12), included in the MPC on-line optimization problem over a finite horizon, involve prediction of future values of process outputs  $y(k+i|k)$ . This prediction is made with the use of a model and is never exact. One way to describe the uncertainty in future output predictions is to consider (a) uncertainty in the model describing the effect of manipulated variables on process outputs, and (b) uncertainty in future disturbances. Both kinds of uncertainty are difficult to capture. For example, model uncertainty may be described as parametric uncertainty or structural uncertainty. Similarly, disturbance uncertainty may be described in terms of a stochastic model, but that model may vary drastically with time. Therefore, quantification of the uncertainty of future output predictions cannot possibly capture all possible cases. In this paper we will focus on a particular case described in detail below.

The future output prediction  $y(k+i|k)$ , made at time  $k$ , is given by the linear model

$$(13) \quad y(k+i|k) = \sum_{j=1}^N h_j u(k+i-j|k) + d(k+i|k)$$

The above model can be used for stable processes, for which it can capture dynamics of any order. The prediction made by the above model has two sources of uncertainty: (a) the uncertain coefficients  $h_j$ , and (b) the uncertain future disturbance  $d(k+i|k)$ . To simplify the presentation, we will assume that  $d(k+i|k) = d(k|k)$ . Moreover, we will assume that  $d(k|k)$  can be estimated as

$$(14) \quad d(k+i|k) = d(k|k) = y(k) - \sum_{j=1}^N h_j u(k-j)$$

In practice, eqn. (14) implies that the disturbance does not contain any high frequency components, or that these frequencies are filtered out in the feedback path. After substitution and rearrangement using the relationships  $s_j = \sum_{i=1}^j h_i$ ,  $j=1,2,\dots$ , between the step and impulse response coefficients  $s_j$  and  $h_i$ ,

respectively, and  $\Delta u(k) \triangleq u(k) - u(k-1)$ , the impacts of future control moves and past control moves are separated, as

$$(15) \quad y(k+i|k) = \sum_{j=1}^i s_j \Delta u(k+i-j|k) + \sum_{j=1}^{N-1} (s_{i+j} - s_j) \Delta u(k-j) + y(k)$$

or in vector/matrix notation,

$$(16) \quad y(k+i|k) = \mathbf{s}^T \mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + \mathbf{s}^T (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p + y(k)$$

where  $\mathbf{s} \in \mathfrak{R}^N$ ;  $\Delta \mathbf{u}_f \in \mathfrak{R}^m$  and  $\Delta \mathbf{u}_p \in \mathfrak{R}^{N-1}$  are the future and past input moves, respectively;  $p < N$ ; and the matrices  $\mathbf{D}_i$ ,  $\mathbf{F}_i$  and  $\mathbf{L}$  are as follows:

$$\mathbf{D}_i = \begin{bmatrix} \mathbf{I}_{i \times i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-i \times N-i} \end{bmatrix} \in \mathfrak{R}^{N \times N}$$

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{0}_{i-1 \times p-i} & \mathbf{0}_{i-1 \times N-1-p+i} \\ 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 & \dots & \dots & 1 \end{bmatrix} \in \mathfrak{R}^{p \times N-1}$$

$$\mathbf{L} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \\ \mathbf{0}_{p-m \times m} \end{bmatrix} \in \mathfrak{R}^{p \times m}$$

If the coefficient vector  $\mathbf{s}$  is a random variable, then, by eqn. (16) the output  $y(k+i|k)$  is also random. Consequently, instead of the constraints in eqn. (12), one has to consider constraints of the form

$$(17) \quad \Pr\{y(k+i|k) \leq y_{\max}\} \geq \alpha, \quad i = 1, \dots, n_c$$

Using eqn. (16) to substitute  $y(k+i|k)$  into the above equation yields

$$(18) \quad \Pr\{\mathbf{s}^T (\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p) \leq y_{\max} - y(k)\} \geq \alpha, \quad i = 1, \dots, n_c$$

The above equation is of the form of eqn. (5). Assuming that  $\mathbf{s}$  is normally distributed with mean  $\bar{\mathbf{s}}$  and covariance  $\mathbf{P}_s$ , one can use eqn. (9) to convert eqn. (18) to its deterministic counterpart

$$(19) \quad \bar{\mathbf{s}}^T (\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p) \leq y_{\max} - y(k) - K_\alpha \sqrt{(\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p)^T \mathbf{P}_s (\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p)}, \quad i = 1, \dots, n_c$$

where the decision variable is the vector  $\Delta \mathbf{u}_f$ .

#### Remarks

- The constraint in eqn. (19) is convex. Indeed, the above constraint can be written as

$$(20) \quad \bar{\mathbf{s}}^T \mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + \bar{\mathbf{s}}^T (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p + y(k) + K_\alpha \left\| \mathbf{P}_s^{0.5} (\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p) \right\|_2 \leq y_{\max}$$

The linear part of the constraint is trivially convex, with all norms being convex as well. Using this form, we show that the constraint is convex, deterministic and is easily incorporated into the standard model predictive control algorithm.

- When the MPC on-line optimization problem becomes infeasible due to excessively tight output constraints, those constraints can be softened through the introduction of additional softening variables, (Zafiriou and Chiou, 1993), as

$$(21) \quad \Pr\{y(k+i|k) \leq y_{\max} + \varepsilon_i\} \geq \alpha, \quad i = 1, \dots, n_c$$

In that case eqn. (18) becomes

$$(22) \quad \Pr\left\{\bar{\mathbf{s}}^T \left( \mathbf{D}_i \mathbf{L} - \mathbf{1} \begin{bmatrix} \Delta \mathbf{u}_f & \varepsilon_i \end{bmatrix}^T + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p \right) \leq y_{\max} - y(k)\right\} \geq \alpha, \quad i = 1, \dots, n_c$$

which, in turn, can be converted to a corresponding deterministic inequality using eqn. (9).

- For an output constrained MPC system employing a process model with uncertain parameters, a simple deterministic constraint of the form

$$(23) \quad \bar{\mathbf{s}}^T (\mathbf{D}_i \mathbf{L} \Delta \mathbf{u}_f + (\mathbf{F}_{i+1} - \mathbf{F}_1) \Delta \mathbf{u}_p) \leq y_{\max} - y(k), \quad i = 1, \dots, n_c$$

is usually formulated. That constraint fails to account for the uncertainty in the process model parameters, captured by the last term in eqn. (19). That term involves the covariance matrix  $\mathbf{P}_s$  of the model parameters, which is usually obtained in standard least-squares identification experiments, along with the parameter estimates.

- One alternative for the enforcement of output constraints in the presence of process model uncertainty would be to simply tighten those bounds, independently of process inputs. However, there are two difficulties with that approach: (a) Process output bounds may have to be made excessively tight, thus making the overall closed-loop system unnecessarily conservative, and (b) To avoid that conservatism, output bounds may not be made tight enough, thus resulting in possible violation of output constraints.

## Case Study

### Process

A continuous-time, 5-state dynamic model of a high-purity distillation process from Skogestad and Postlethwaite (1996), sampled at a rate of 2 minutes, was used to analyze the chance constraint formulation. The purity in the two output streams was to be controlled by reflux ratio and boil-up rate in an L/V feedback control configuration as shown in Figure 1.

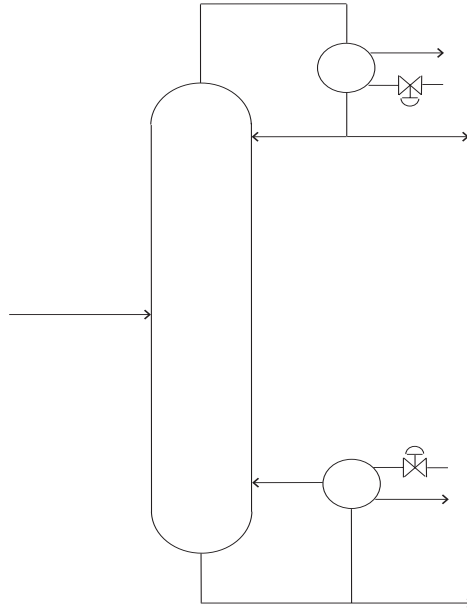


Figure 1

The process model is nearly singular resulting in poorly conditioned output constraints. The Hessian of the quadratic MPC on-line objective is also poorly conditioned resulting in a challenging optimization problem.

### Identification

The system used in the simulations is a multivariable system with 2 inputs and 2 outputs, with  $n_c$  chance constraints applied to each individual output. Thus there is a constraint softening term for each output at each of the  $n_c$  predictions into the future, yielding a total of  $2n_c$  constraint softening terms. Each output is predicted on the basis of the two FIR models corresponding to the two inputs. Using a vector representation, the model is arranged as follows:

$$(24) \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{s}_{11}^T & \mathbf{s}_{12}^T \\ \mathbf{s}_{21}^T & \mathbf{s}_{22}^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

To develop a process model and uncertainty description for the purpose of demonstrating our method, we generated output data from the aforementioned state space model using a pseudo random binary sequence (PRBS) input with an amplitude of 1. The output data was then corrupted with normally distributed noise with a given signal to noise ratio between 11 and 18. Standard least-squares techniques were then used to identify multi-input-single-output (MISO) finite impulse response (FIR) models for each output using the corrupted data.

Three models were generated which show the benefits of chance constrained model predictive control. Each model was identified using a different number of input-output data points: 200, 500 and 1000; and similar signal-to-noise ratios, yielding models with differing levels of uncertainty. Figure 2, Figure 3, and Figure 4 show the impulse response coefficients and their corresponding covariance matrices. Each of these models was then utilized in the creation of two model predictive controllers, the first corresponding to standard output constrained MPC, and the second corresponding to MPC with chance constraints.

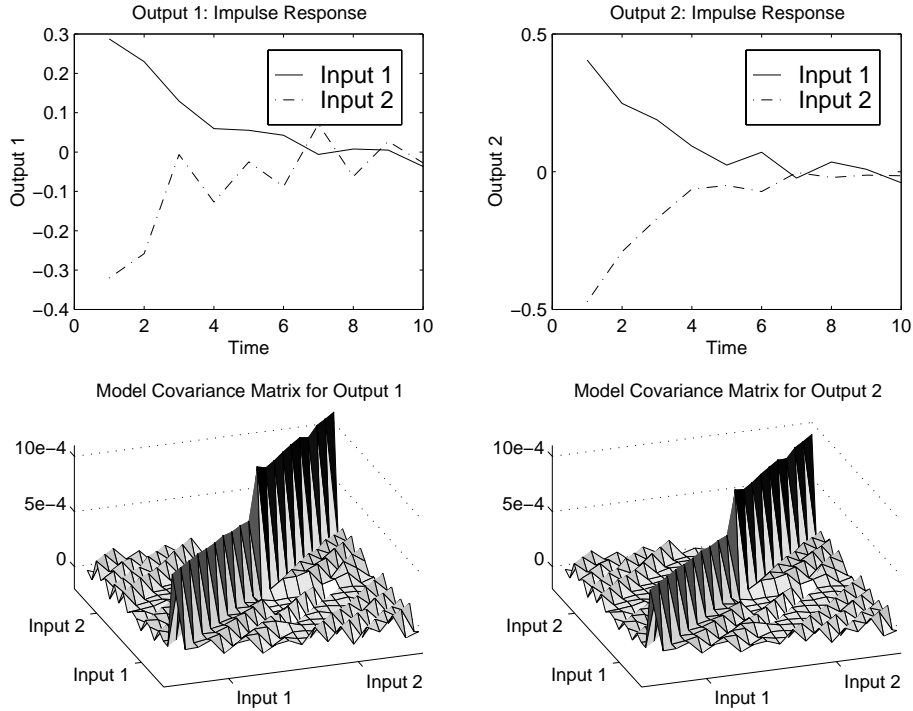


Figure 2. 200 I/O Points: 15, 18 SNR

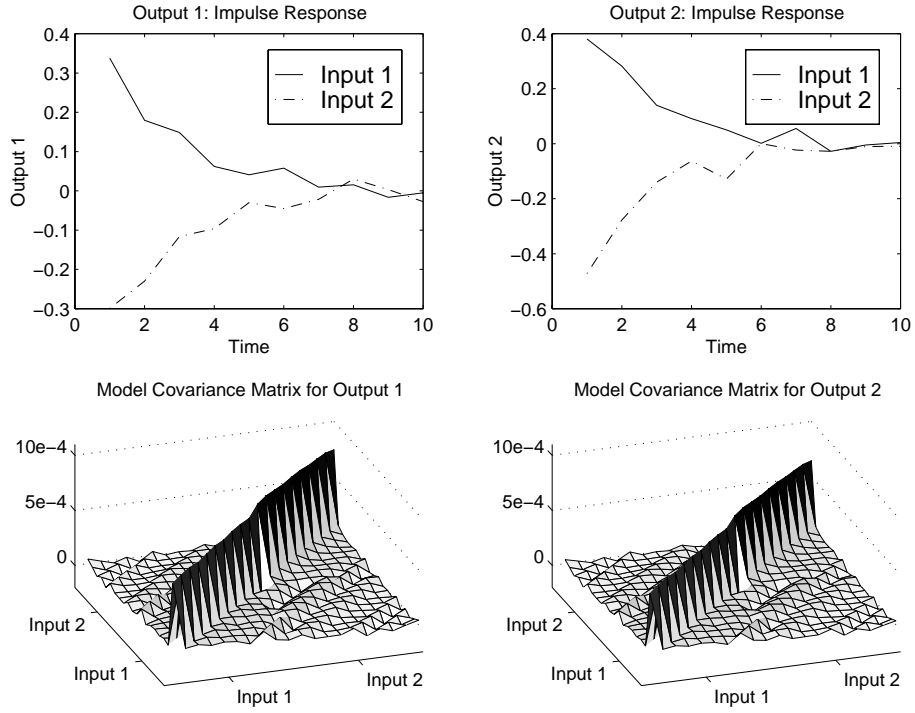


Figure 3. 500 I/O Points: 13, 16 SNR

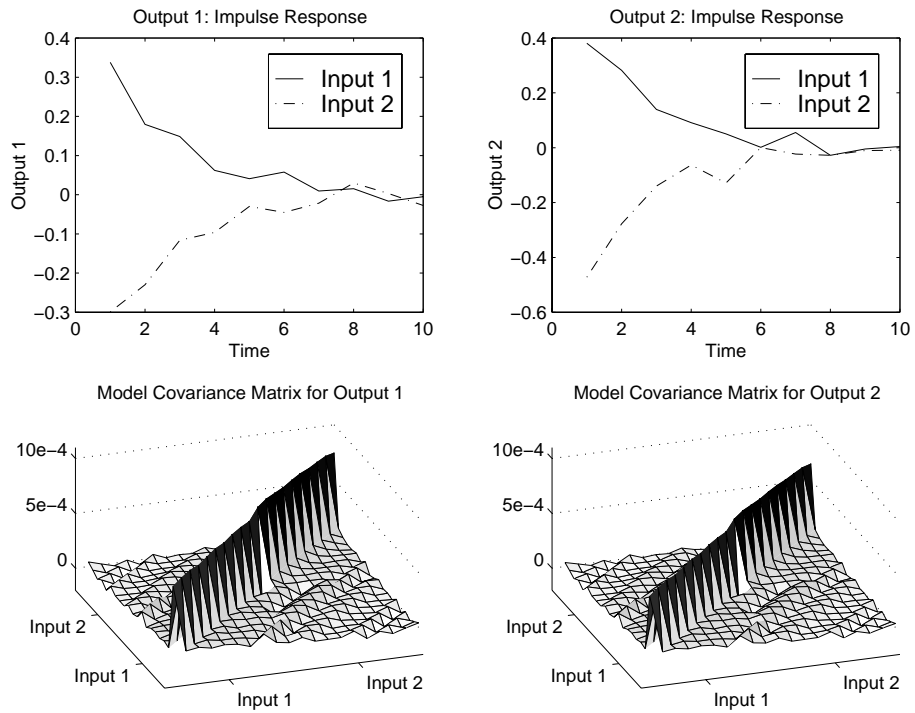


Figure 4. 1000 I/O Points: 11, 13 SNR



### Chance Constraints

The directions of maximum uncertainty for the three models can be determined from the following optimization problem, according to eqn. ( 19 ):

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{x}^T \mathbf{P} \mathbf{x} \\ \text{s.t.} \quad & -\mathbf{1} \leq \mathbf{x}(i) \leq \mathbf{1}, \quad i = 1,2 \end{aligned}$$

Each model identified used a different Hessian for this mathematical programming problem:

$$\mathbf{P}_{200} = \begin{bmatrix} 7.65 \cdot 10^{-4} & -2.95 \cdot 10^{-5} \\ -2.95 \cdot 10^{-5} & 1.14 \cdot 10^{-3} \end{bmatrix}$$

$$\mathbf{P}_{500} = \begin{bmatrix} 6.70 \cdot 10^{-4} & 4.83 \cdot 10^{-5} \\ 4.83 \cdot 10^{-5} & 7.93 \cdot 10^{-4} \end{bmatrix}$$

$$\mathbf{P}_{1000} = \begin{bmatrix} 5.28 \cdot 10^{-4} & 2.44 \cdot 10^{-6} \\ 2.44 \cdot 10^{-6} & 5.41 \cdot 10^{-4} \end{bmatrix}$$

The problem to be solved is the maximization of a convex function over a convex set (equivalent to the minimization of a concave function over a convex set) which, in general, is *NP-hard* (Horst and Tuy, 1990). The solutions of such problems are always found on the boundary of the feasible region. For a problem with very small dimensionality and symmetry, this problem does not present any significant difficulty. The problem can be solved using standard constrained nonlinear optimization techniques and the symmetry of the constraints and objective can be used to determine any degenerate solutions to the problem. The optimization gives solutions at the four corners of the feasible region: (1,1), (1,-1), (-1,1) and (-1,-1); depending on the starting point of the optimization problem. By inspecting the values of the objective function at these points, the true optima can be determined. The first objective,  $\mathbf{x}^T \mathbf{P}_{200} \mathbf{x}$ , predicts the optima to be at the points (1,-1) and (-1,1). The two subsequent objectives show the optima to be at the points (1,1) and (-1,-1). Due to the gains and the ill-conditioning of the system, the optima for the latter two objectives agree with the nature of the system. This can be seen in Figures 2, 3 and 4. Since the cross-correlation for the variables are rather small, the smallest data set actually estimated the cross-correlation to be negative, thus the change in direction is due to the stochastic nature of the identification.

### Results

Results for the three models are shown in Figure 5, Figure 6, and Figure 7. These figures show the closed-loop responses of the two outputs (top and bottom temperatures) to setpoint step change in the top temperature, output 1, of magnitude  $-5 \times 10^{-3}$ . The bottom temperature, output 2, was constrained by a lower bound of  $-1 \times 10^{-3}$  and for the chance constrained MPC the probability of constraint violation was set to 0.01. Figure 8 shows that application of standard MPC to that system resulted in frequent and large violations of the output bound. In contrast, application of chance-constrained MPC succeeded in preventing violation of the output bound. All simulations were accomplished in a Matlab environment using the Numerical Analysis Group (NAG) nonlinear constrained optimizer and the controller described in Table 1.

Table 1

MPC parameter	Control horizon $m$	Input move weight $r$	Prediction horizon $p$	Output constraint softening weight $q$	Confidence level, $\alpha$
MPC terms	$r \sum_{i=0}^{m-1} \Delta u(k+i k)^2$		$\sum_{i=1}^p (y(k+i k) - y^{SP})^2$	$\Pr \left\{ \begin{matrix} y_{\min} - \epsilon \leq y \\ \leq y_{\max} + \epsilon \end{matrix} \right\} \geq \alpha$ $q\epsilon^2$	eqn. (1)
Standard output constrained MPC	4	0.02	14	$10^3$	n/a
Chance Constrained MPC	4	0.02	14	3	99%

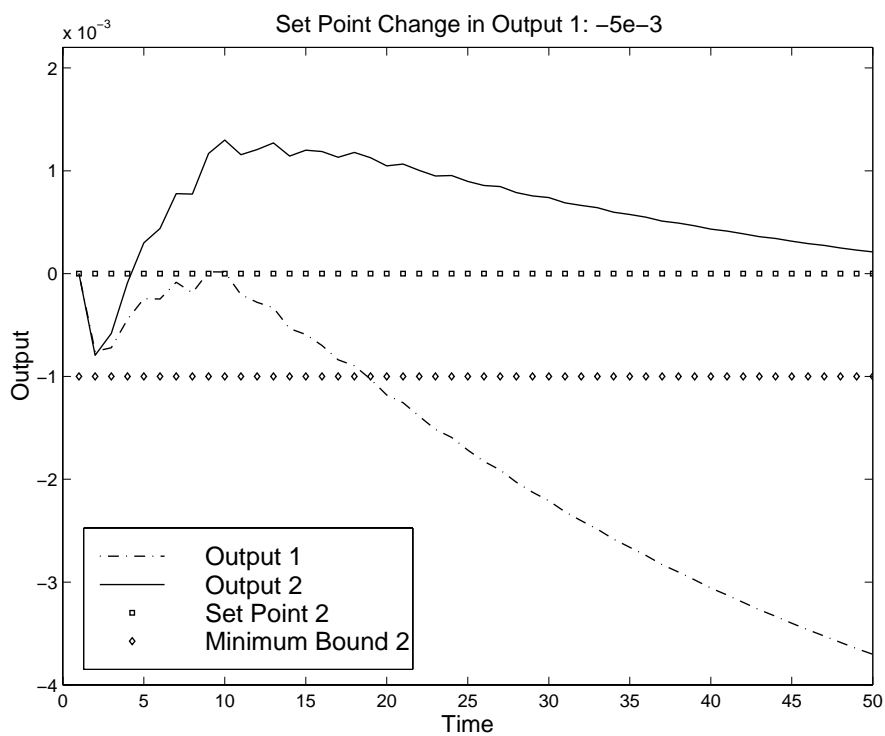


Figure 5. 200 I/O points

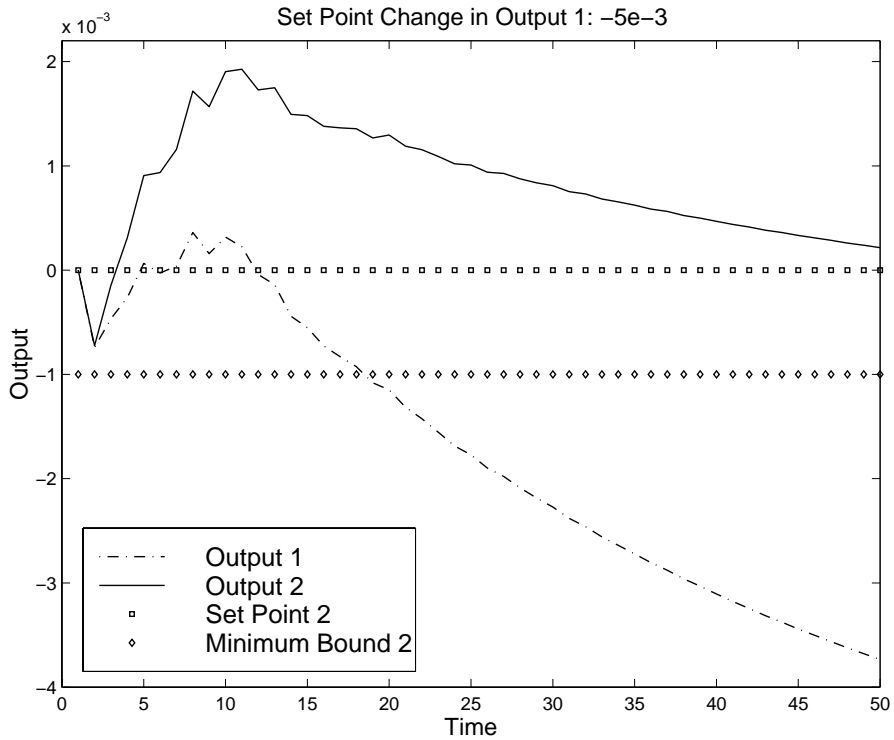


Figure 6. 500 I/O points

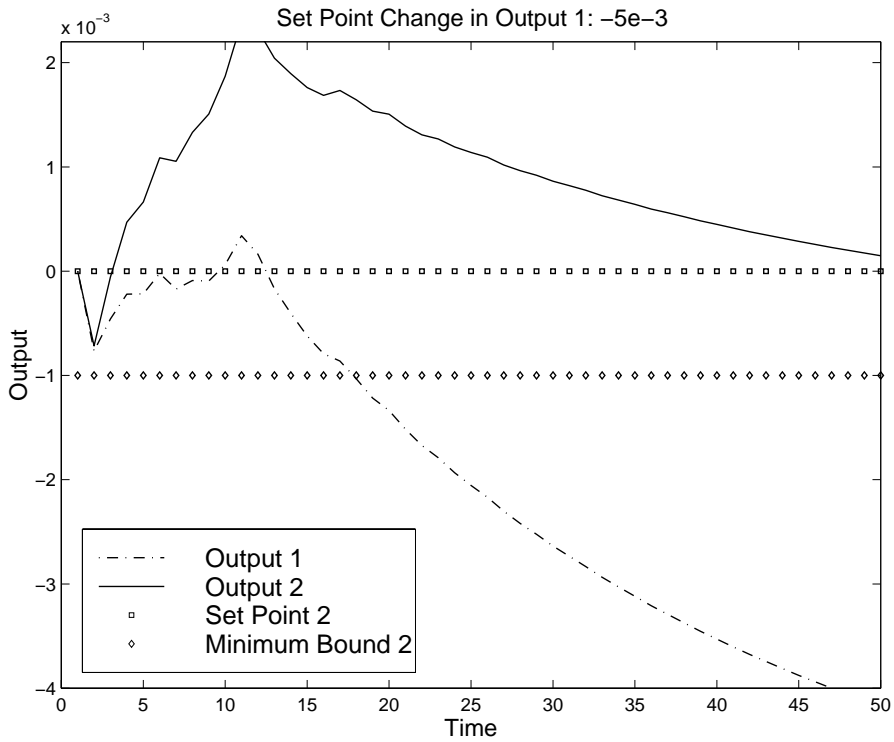


Figure 7. 1000 I/O points

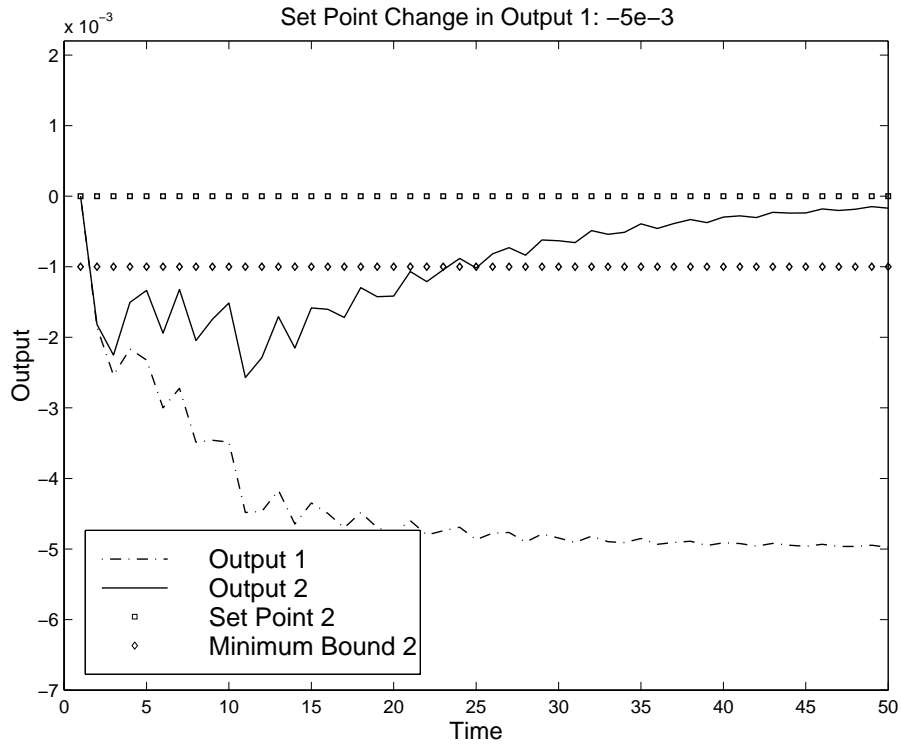


Figure 8: 200 I/O points

One way of attempting to deal with the problems associated with standard output constrained MPC is to tighten the bounds placed on the process outputs. The following figures show how this procedure can still lead to output bounds violation. Process outputs are predicted based on the process model. Since this model is uncertain, the output can be predicted to be within bounds while in actuality the output violates the bounds. In the case examined here, the bound is close to the setpoint so that even placing the bound above the set point would cause offset.

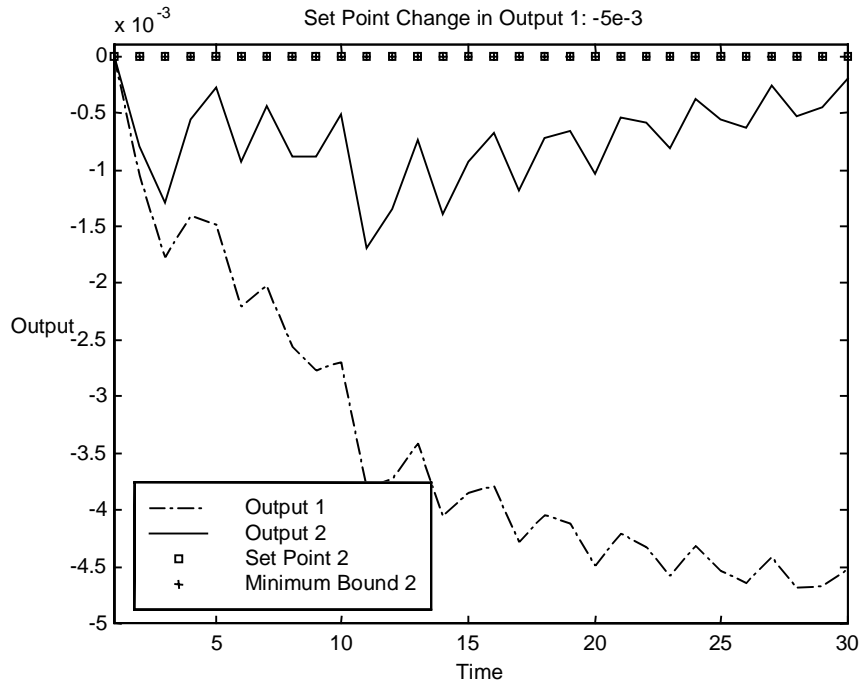


Figure 9. 200 I/O Points

## Discussion and future research

In this work, we focused on robustness of constrained MPC with respect to satisfaction of process output constraints by a closed-loop MPC system that employs an uncertain process model. We proposed to enhance MPC robustness by formulating process output constraints as chance constraints. Simulations comparing the performance of the chance-constrained MPC formulation versus that of standard MPC with output constraints showed the ability of the proposed approach to improve the robustness of MPC with respect to output constraint satisfaction. For the high-purity distillation column, a nearly singular system, studied in the simulations the original output constraints were violated frequently and to a large degree when standard MPC was used, even with extremely large penalties on the constraint softening variable  $\epsilon$ . With the use of chance constraints, violation of output constraints was drastically reduced or eliminated.

While chance output constraints were used in this work within a standard MPC framework, there are several possibilities for using chance constraints in other control frameworks, such as stabilization of uncertain processes or combined MPC and identification where process modeling uncertainty can be naturally incorporated in the on-line optimization problem. In addition, uncertainty in future output predictions due to stochastic disturbances may also be incorporated in the proposed framework.

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